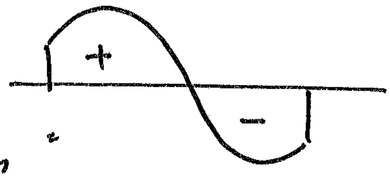
8 april 2015

Buelengde, areal og volum

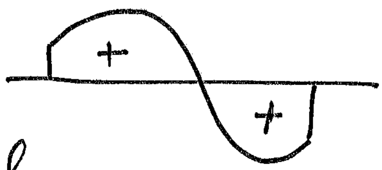
1

$$\int_a^b f(x)\,dx$$

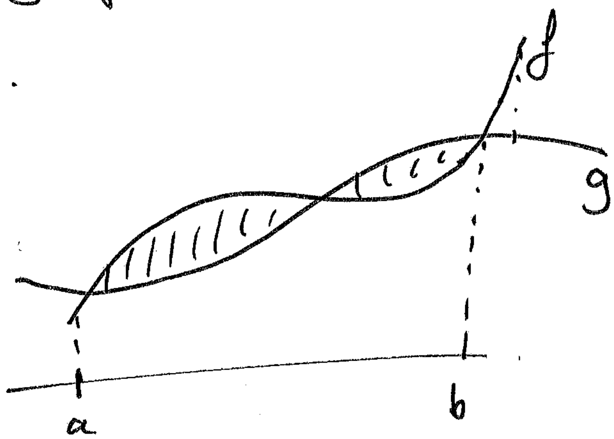


"areal med fortegn"

$$\int_a^b |f(x)|\,dx$$

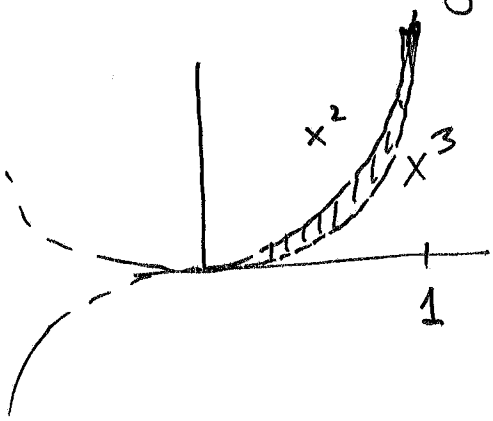


arealet mellom grafen til $f(x)$ og x-aksen.



Arealet til regionen mellom grafen til f og g. $\int_a^b |f-g|\,dx$

Eksempel Finn arealet som er avgrenset av grafen til x^2 og grafen til x^3



$$A = \int_0^1 x^2 - x^3\,dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4}$$

$$= \underline{\underline{\frac{1}{12}}}$$

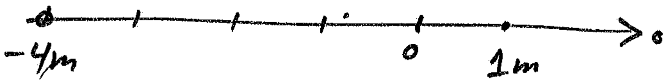
Posisjonen til en bil fra $t = 0s$ til $t = 2s$

(2) er gitt ved $s(t) = 3\frac{m}{s^2} \cdot t^2 - 2\frac{m}{s^3} t^3$.

$S(0) = 0$

$s(2s) = 3\frac{m}{s^2}(4s^2) - \frac{2m}{s^3} \cdot (8s^3)$

$= -4m$



Hvor langt kjører bilen de to første sekundene.

$$L = \int_0^{2s} |V(t)| \, dt$$

$$V(t) = S'(t) = 6\frac{m}{s^2} \cdot t - 6\frac{m}{s^3} \cdot t^2 \; (= 6t(1-t))$$

$V(t) \geq 0$ for $0 \leq t \leq 1s$

$V(t) \leq 0$ for $1s \leq t \leq 2s$.

$$L = \int_0^{1s} V(t)\,dt - \int_{1s}^{2s} V(t)\,dt$$

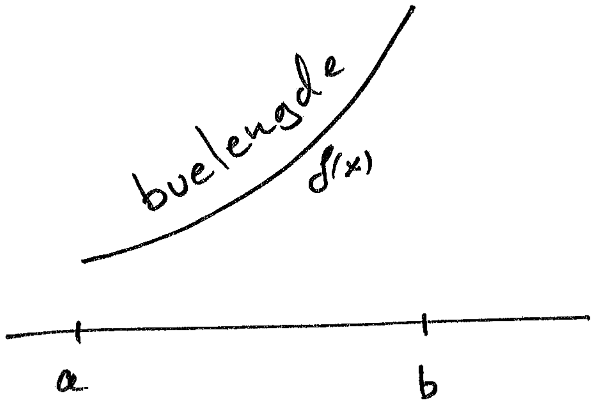
$$s(t)\Big|_0^{1s} - s(t)\Big|_{1s}^{2s}$$

$$= S(1s) - S(0s) - (s(2s) - S(1s))$$

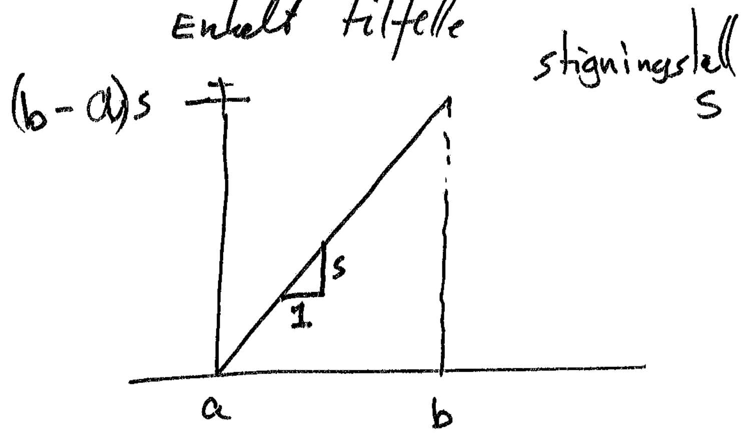
$$= 1m - 0m - (-4m - 1m)$$

$$= \underline{\underline{6m}}$$

③



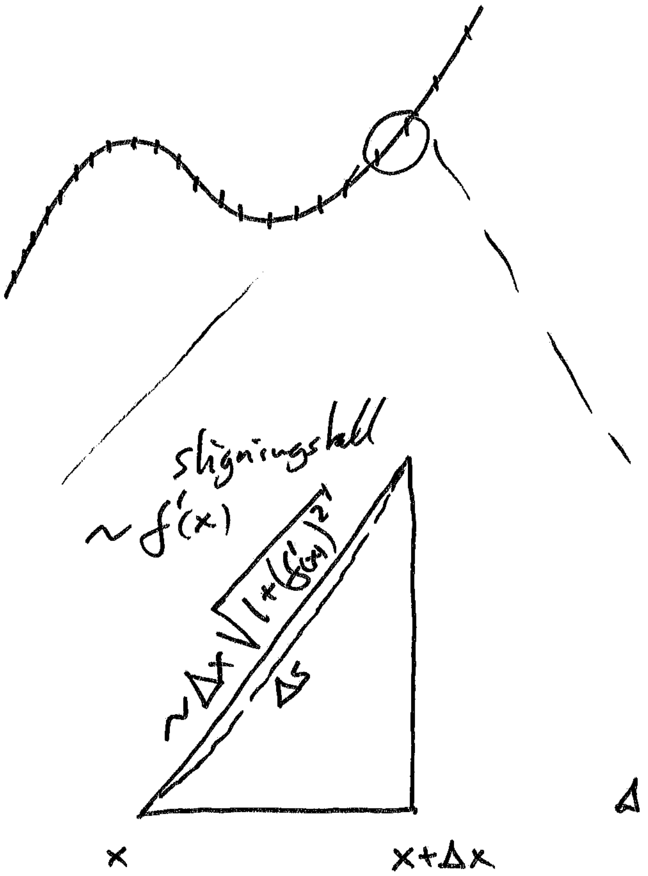
Enkelt tilfelle



Pytagoras :

Buelengden (lengden til den skrå linjen) er

$$\sqrt{(b-a)^2 + (s(b-a))^2} = (b-a)\sqrt{1+s^2}$$



Buelengden

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

$\Delta x \to 0.$

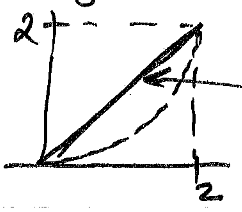
$$\Delta s \sim \sqrt{1+(f'(x))^2}\,\Delta x$$

eks : $f(x) = x^2/2 \qquad f'(x) = x$ Buelengde til parabelen fra $x=a$ til $x=b$: $\int_a^b \sqrt{1+x^2}\, dx$

$$\left(= \frac{1}{2}\left(x\sqrt{1+x^2} + \operatorname{arcsinh}(x) \right)\Big|_a^b \right)$$

Numerisk int. gir $\int_0^2 \sqrt{1+x^2}\, dx = 2.9578\ldots$

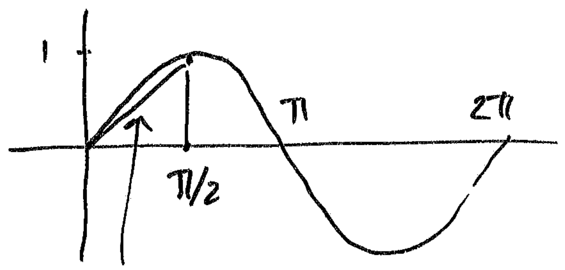
Enkelt estimat



lengden til linjen er : $2\sqrt{2} = 2.82\ldots$

Hva er lengden på sinus kurven fra 0 til 2π?

4



lengden på linjen er $\sqrt{1+\left(\frac{\pi}{2}\right)^2}$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$L = \int_0^{2\pi} \sqrt{1+\cos^2 x}\, dx$$

$$1 \cdot 2\pi < L \leq \sqrt{2} \cdot 2\pi \sim 8.885$$

$$4\sqrt{1+\left(\frac{\pi}{2}\right)^2} < L$$

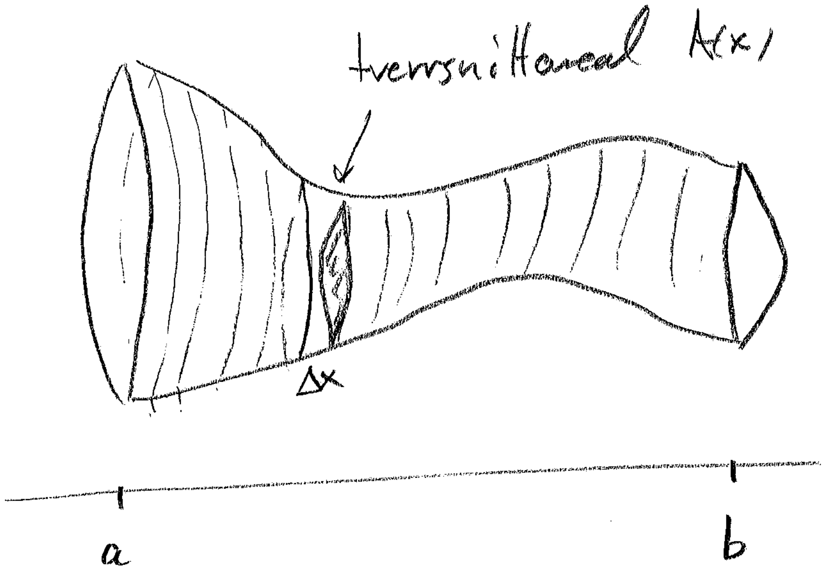
2

7.448...

Numerisk integrasjon gir $\underline{L = 7.64039....}$

Volum ved skivemetoden

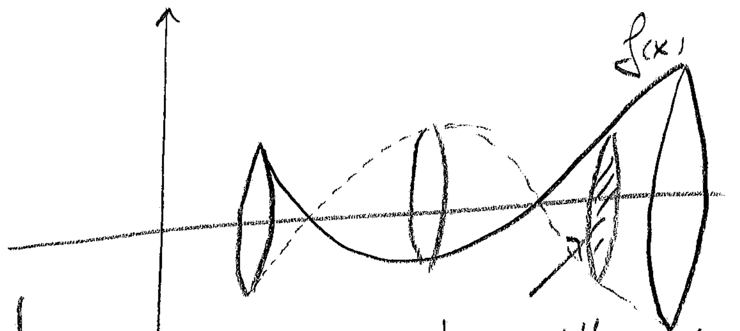
5



$$V \sim \sum A(x_i)\cdot \Delta x_i \qquad \text{Riemann sum} \qquad \text{alle } \Delta x_i \to 0 \text{ (antall oppdelinger) } \to \infty$$

$$V = \int_a^b A(x)\,dx \qquad \text{Riemann integral}$$

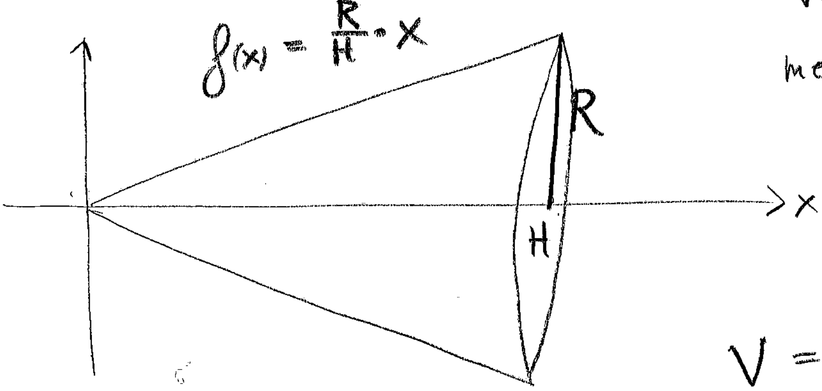
Rotasjonslegemer



Volumet til rotasjonslegemet som fremkommer av å rotere regionen begrenset av grafen til $f(x)$ for $a \le x \le b$ rundt x-aksen er

tverrsnittarealet
$A(x) = \pi\,|f(x)|^2$
$= \pi\cdot(f(x))^2$

$$V = \pi\int f(x)^2\,dx$$



Volumet til en kjegle med høyde H og radius R

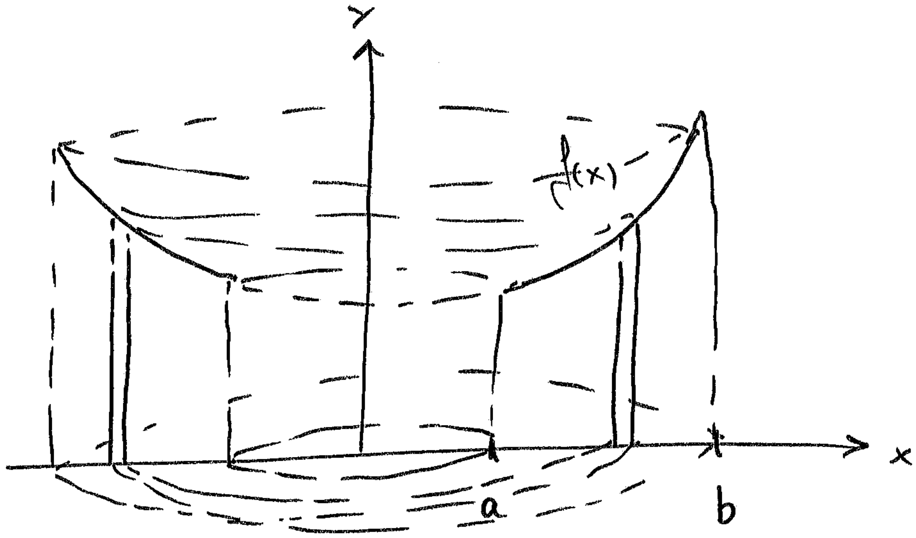
$$V = \pi\int_0^H \left(\frac{R}{H}x\right)^2 dx$$

$$= \pi\frac{R^2}{H^2}\,\frac{x^3}{3}\Big|_0^H = \frac{\pi\cdot R^2\cdot H^3}{3\cdot H^2}$$

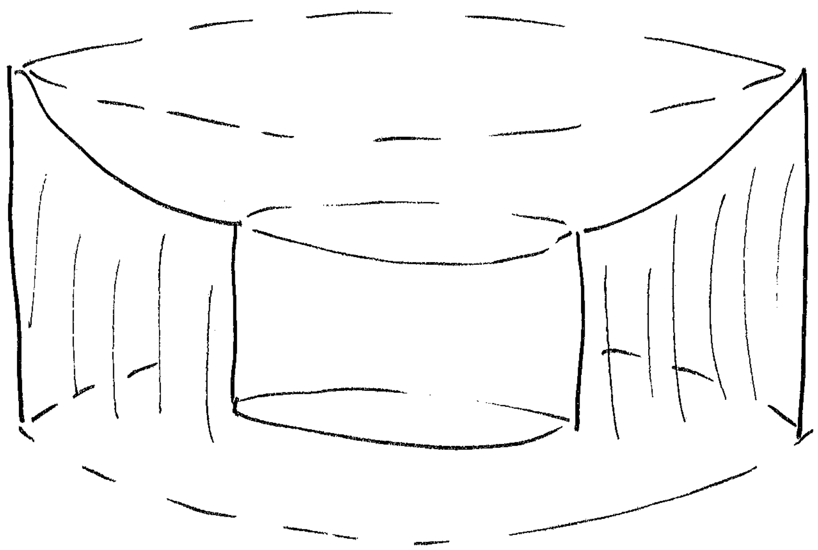
$$V = \frac{1}{3}(\pi R^2)H$$

Skallmetoden

6

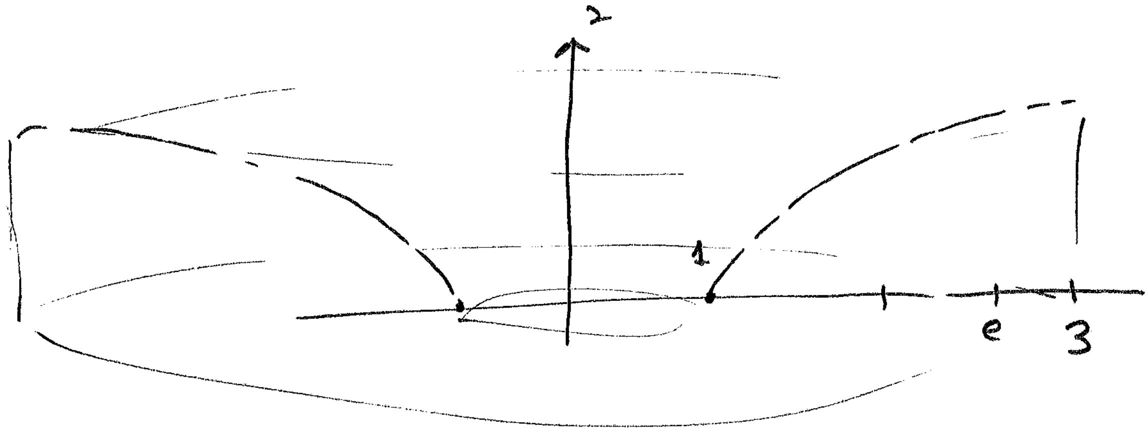


$$\Delta V \sim \Delta x\, |f(x)|\, 2\pi \cdot x$$



$$V = \int_a^b |f(x)| \cdot 2\pi x \, dx = 2\pi \int_a^b |f(x)|\, x \, dx$$

Eksempel



$f(x) = \ln x \qquad 1 \leq x \leq 3$

(7) Volumet til legemet som fremkomme av å rotere regionen mellom grafen til $f(x) = \ln x \quad 1 \leq x \leq 3$ og x-aksen, rundt y-aksen er:

$$V = 2\pi \int_1^3 \underbrace{\ln(x)}_{u} \cdot \underbrace{x}_{v'} \, dx \qquad \begin{array}{l} v' = x \\ v = \frac{x^2}{2} \end{array}$$

delvis integrasjon

$$= 2\pi \left[\frac{x^2}{2} \ln x \Big|_1^3 - \int_1^3 \overset{u'}{\frac{1}{x}} \cdot \overset{v}{\frac{x^2}{2}} \, dx \right]$$

$$= 2\pi \left[\frac{x^2}{2} \ln x \Big|_1^3 - \frac{x^2}{4} \Big|_1^3 \right]$$

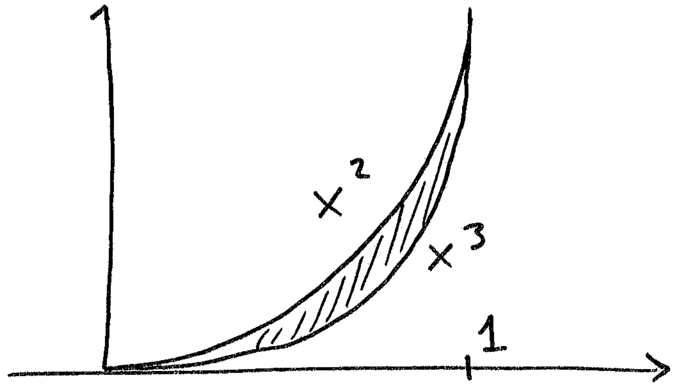
$$= 2\pi \, \frac{x^2}{4} (2 \ln x - 1) \Big|_1^3$$

$$= 2\pi \left[\frac{9}{4} (2 \ln 3 - 1) - \frac{1}{4} (\overset{0}{\widehat{2 \ln(1)}} - 1) \right]$$

$$= 2\pi \left[\frac{9}{2} \ln 3 - \frac{9}{4} + \frac{1}{4} \right]$$

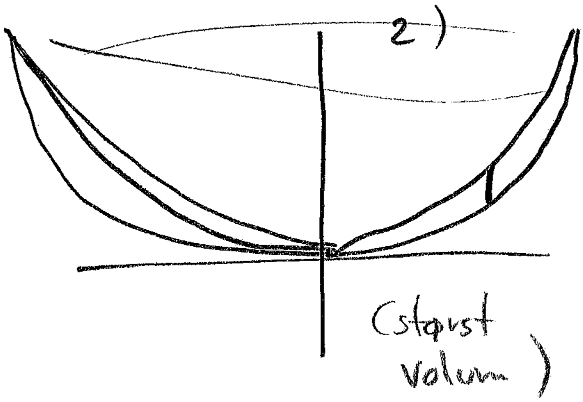
$$= \underline{\pi (9 \ln 3 - 4)}$$

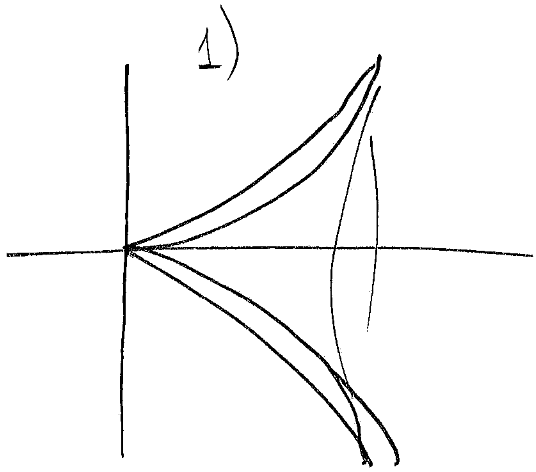
(8)



Roter regionen rundt 1) x-aksen
2) y-aksen

Finn volumet i begge tilfeller.





1) $$\pi \int_0^1 (x^2)^2 \, dx - \pi \int_0^1 (x^3)^2 \, dx$$

$$= \pi \left[\frac{x^5}{5} \Big|_0^1 - \frac{x^7}{7} \Big|_0^1 \right] = \pi \left(\frac{1}{5} - \frac{1}{7} \right)$$

$$= \underline{\underline{\frac{2\pi}{35}}} \quad \left(= \frac{4}{7} \cdot \frac{\pi}{10} \right)$$

2) $$\int_0^1 2\pi |x^2 - x^3| \cdot x \, dx$$

$$= 2\pi \int_0^1 x^3 - x^4 \, dx = 2\pi \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1$$

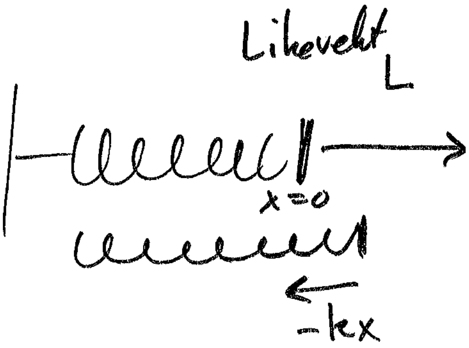
$$= 2\pi \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{2\pi}{20} = \underline{\underline{\frac{\pi}{10}}}$$

Arbeid = Kraft · veg.

(9) $\xrightarrow{F(x)}$

$$W = \int F(x)\,dx$$

Fjær



Arbeid utført ved å dra en fjær med fjærstivhet k fra jamvekt posisjon ut en lengde L

$$W = \int_0^L kx\,dx = \underline{\tfrac{1}{2}kL^2}$$

Kinetisk energi

$\xrightarrow{V(t)}$

$F = mV'(t)$

$$\Delta s = V(t)\cdot\Delta t$$

$$W = \int F\cdot\frac{ds}{dt}\,dt$$

$$\int m\cdot\underbrace{V'\cdot V}_{\left(\frac{V^2}{2}\right)'}\,dt$$

subst.

$$= \int mV\,dV$$

$$= \tfrac{1}{2}mV^2\Big|_{V_{start}}^{V_{stopp.}}$$

Arbeidet som kreves for å få et objekt med masse m opp i fart V er lik $\underline{\frac{m}{2}V^2}$