(1)

Derivasjonsreglene

$$(f+g)' = f' + g'$$

c konstant $\quad (cf)' = c \cdot f'$ — derivasjon er lineær

$$(f \cdot g)' = f' \cdot g + f \cdot g' \quad \text{(produktregelen)}$$

$$(f(u(x)))' = f'(u(x)) \cdot u'(x) \quad \text{(kjerneregelen)}$$

$$\frac{d\, f \circ u}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Eksempler: * $(3\sin x - \pi \cdot \cos x + 2)'$

$$= (3 \cdot \sin x)' + (-\pi \cos x)' + (2)'$$

$$= 3(\sin x)' - \pi (\cos x)' + (2)'$$

$$= 3\cos x - \pi(-\sin x) + 0 = \underline{3\cos x + \pi \sin x}$$

* $f(x) = (x-1)^4 (x+2)^7$ — mye arbeid å gange ut! Ungår det.

$$f'(x) = \underbrace{((x-1)^4)'}_{4(x-1)^3 \cdot (x-1)'} (x+2)^7 + (x-1)^4 \cdot \underbrace{((x+2)^7)'}_{7(x+2)^6 (x+2)'}$$

$$= 4(x-1)^3 (x+2)^7 + 7(x-1)^4 \cdot (x+2)^6$$

$$= (x-1)^3 (x+2)^6 [4(x+2) + 7(x-1)]$$

$$= \underline{(x-1)^3 (x+2)^6 (11x + 1)}$$

* $x^2 \cdot \sin x$

$$(x^2 \cdot \sin x)' = (x^2)' \sin x + x^2 (\sin x)'$$

$$= \underline{2x \sin x + x^2 \cos x}$$

$*$ $f(x) = \sin^4(x) = (\sin(x))^4$

(2) $g(x) = \sin(x^4)$

$$f'(x) = 4(\sin(x))^3 \cdot (\sin x)'$$
$$= \underline{4\sin^3(x) \cdot \cos x}$$

$$g'(x) = \underline{\cos(x^4) \cdot 4x^3}$$

$h(u) = u^4$

$f(x) = h(\sin x)$
$g(x) = \sin(h(x))$

$*$ $f(x) = \sqrt{2x}$

$$f(x) = \sqrt{2} \cdot \sqrt{x}$$
$$f'(x) = \sqrt{2} \cdot (\sqrt{x})' = \sqrt{2}(x^{1/2})'$$
$$= \sqrt{2} \cdot \frac{1}{2} x^{-1/2} = \frac{\sqrt{2}}{2} \frac{1}{\sqrt{x}}$$
$$= \frac{1}{\sqrt{2}\sqrt{x}} = \frac{1}{\sqrt{2x}}.$$

Vi observerer at

$f(x) \cdot f'(x) = 1 \quad (x>0)$

$*$ $f(x) = e^{(e^{-x^2})}$

Vi anvender kjerneregelen 2 ganger:

$$f'(x) = e^{(e^{-x^2})} \cdot (e^{-x^2})' = e^{(e^{-x^2})}(e^{-x^2}) \cdot (-x^2)'$$
$$= \underline{-2x\, e^{e^{-x^2}} \cdot e^{-x^2}}.$$

$*$ Linear substitusjon (kjernen er en lineær funksjon)

$$\underline{\frac{d}{dx} f(ax+b) = af'(ax+b)}$$

Minner om definisjon av den deriverte

(3) $f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$.

$\Delta f \sim f'(x) \cdot \Delta x$ lineær tilnærming.

$$(f+g)' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - (f(x) + g(x))}{\Delta x} = \lim_{\Delta x \to} \frac{\Delta(f+g)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f + \Delta g}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \frac{\Delta g}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = f'(x) + g'(x)$$

$(c \cdot f)'$ tilsvarende.

produktregelen: $(f \cdot g)(x+\Delta x) = f(x+\Delta x) \cdot g(x+\Delta x)$

$$(f(x) + \Delta f)(g(x) + \Delta g) = f(x) \cdot g(x) + \Delta f \cdot g(x) + f(x)\,\Delta g + \Delta f \cdot \Delta g$$

$$\frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x} = \frac{(f(x) + \Delta f)(g(x) + \Delta g) - f(x)\,g(x)}{\Delta x}$$

$$= \frac{\Delta f}{\Delta x} \cdot g(x) + f(x) \cdot \frac{\Delta g}{\Delta x} + \frac{\Delta f \cdot \Delta g}{\Delta x}.$$

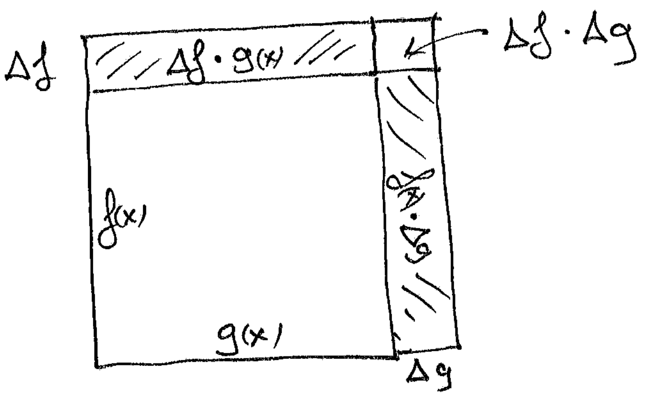
Tar vi grensen $\Delta x \to 0$ får vi

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$\left(\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \Delta g = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta g = f'(x) \cdot 0 = 0 \right)$$

(antar $f(x), g(x)$, $\Delta f, \Delta g > 0$)

geometrisk:



Kjerneregelen:

(4)

$$\Delta(f\, u)(x) = f(\underbrace{u(x+\Delta x)}_{\sim u(x) + u'(x)\Delta x}) - f(u(x))$$

$$f(u(x) + u'(x)\Delta x) - f(u(x))$$

$$\sim f(u(x)) + f'u(x) \cdot (u'(x)\Delta x) - f(u(x))$$

$$= \underline{\underline{f'(u(x)) \cdot u'(x)}}\ \Delta x.$$

$\frac{1}{x}$ Bruker def. til å derivere $\frac{1}{x}$

$$\lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\frac{x - (x+\Delta x)}{(x+\Delta x)\cdot x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} \cdot \frac{1}{(x+\Delta x)\cdot x} = \frac{-1}{x^2}.$$

Legg merke til: $\left(\frac{1}{x}\right)' = (x^{-1}) = (-1) \cdot x^{-1-1}$

(følger formelen $x^r = r x^{r-1}$) $= \frac{-1}{x^2}.$

$$(x^{-7}) = (x^7)^{-1} = \frac{1}{x^7}.$$

Kjerneregelen: $(x^{-7})' = \frac{-1}{(x^7)^2} \cdot (7x^6) = \underline{-7x^{-8}}.$

Tilsvarende $(x^n)' = n \cdot x^{n-1}$ for alle heltall n.

Vi viser: $(x^{p/q})' = \frac{p}{q} \cdot x^{\frac{p}{q}-1}$

$$(x^{p/q})' = ((x^p)^{1/q})' = \frac{1}{q}(x^p)^{\frac{1}{q}-1} \cdot (x^p)'$$

$$= \frac{1}{q} x^{\frac{p}{q}-p} \cdot p x^{p-1} = \frac{p}{q} x^{\frac{p}{q}-p+p-1} = \underline{\frac{p}{q} x^{\frac{p}{q}-1}}$$

(5) $$x^r \overset{\text{def}}{=} \lim_{\frac{p}{q} \to r} x^{p/q}$$

Derivasjons formelen $$\boxed{\frac{d}{dx} x^r = r \cdot x^{r-1}}$$

er gyldig for alle reelle tall r
(når x^r og x^{r-1} er definert)

(det følger ved å forsikre seg om at grensen $\frac{p}{q} \to r$ ovenfor respekterer derivasjon.)

$$(x^{2/3})' = \frac{2}{3} x^{2/3 - 1} = \underline{\frac{2}{3} x^{-1/3}}$$

$$(x^{\pi})' = \pi x^{\pi - 1}.$$

Kvotientregelen: $$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}.$$

uttedes:

$$\frac{f}{g} = f \cdot \frac{1}{g} = f \cdot (g)^{-1}$$

$$\left(\frac{f}{g}\right)' \overset{\text{prod. regel}}{=} f' \cdot (g)^{-1} + f \underbrace{((g)^{-1})'}_{\frac{-1}{g^2} \cdot g'}$$

$$= \frac{f'}{g} - \frac{f \cdot g'}{g^2}$$

$$= \underline{\underline{\frac{f' \cdot g - f g'}{g^2}}}$$

Eksempel $\dfrac{e^x}{\sqrt{2x-1}}$

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Kvotientregelen: $\dfrac{(e^x)'\sqrt{2x-1} - e^x(\sqrt{2x-1})'}{2x-1}$

$$= \frac{e^x\sqrt{2x-1} - e^x\left(\frac{1}{2}(2x-1)^{-1/2}\cdot 2\right)}{2x-1}$$

$$= \frac{e^x\left(\sqrt{2x-1} - 1/\sqrt{2x-1}\right)}{(2x-1)} = e^x\left(\frac{1}{\sqrt{2x-1}} - \frac{1}{(2x-1)^{3/2}}\right)$$

Alternativt: $\dfrac{e^x}{\sqrt{2x-1}} = e^x\cdot(2x-1)^{-1/2}$

Benytter produktregelen: $\left(e^x\cdot(2x-1)^{-1/2}\right)'$

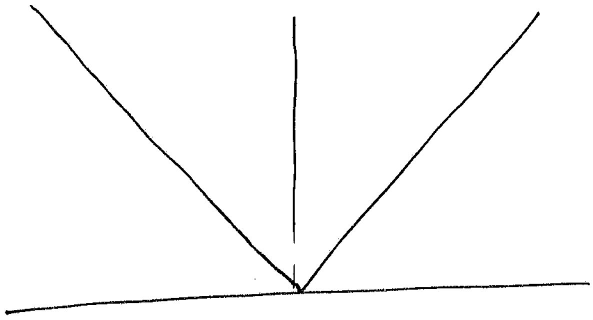
$$= (e^x)'(2x-1)^{-1/2} + e^x\left((2x-1)^{-1/2}\right)'$$

$$= e^x(2x-1)^{-1/2} + e^x\left(-\frac{1}{2}(2x-1)^{-\frac{1}{2}-1}\overbrace{(2x-1)'}^{2}\right)$$

$$= \underline{e^x\left((2x-1)^{1/2} - (2x-1)^{-3/2}\right)}.$$

Hva er $\dfrac{d}{dx}|x|$?

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$\frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

den deriverte eksisterer ikke i 0.

$$\lim_{h\to 0}\frac{|h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\begin{cases} 1 & h>0 \\ -1 & h<0 \end{cases}$$ eksisterer ikke

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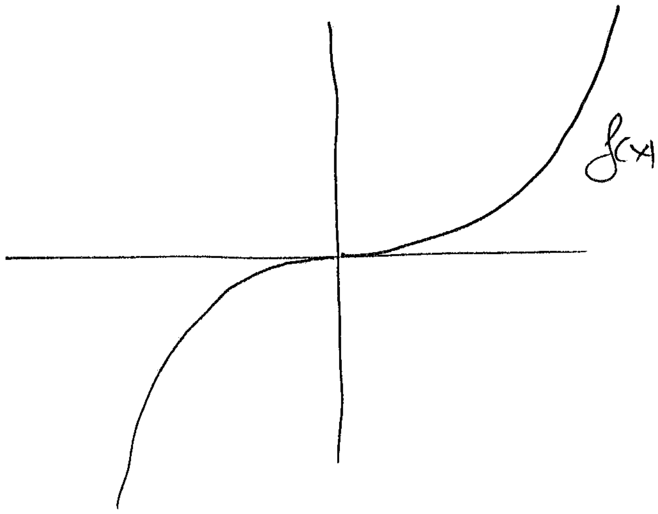
$$f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$



$$f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ -2x & x < 0 \end{cases}$$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

$$= \lim_{h \to 0} \begin{cases} h & h > 0 \\ -h & h < 0 \end{cases} = 0.$$

Kvotientregel eksempel:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \underline{\frac{1}{\cos^2 x}} \quad (= \sec^2 x) = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cancel{\sin}\cos^2 x}$$

$$= \underline{1 + \tan^2 x}$$

$$\underline{(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x}$$

L'Hopitals regel.

(8) Hvis $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ og $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ eksisterer, da eksistere $\lim_{x\to a} \frac{f(x)}{g(x)}$

og $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

Eksempler: * $\lim_{x\to 0} \frac{e^x - 1}{3x}$ type $\frac{0}{0}$

$$\lim_{x\to 0} \frac{(e^x-1)'}{(3x)'} = \lim_{x\to 0} \frac{e^x}{3} = \frac{1}{3}$$

så ved L'Hopital $\lim_{x\to 0} \frac{e^x - 1}{3x} = \frac{1}{3}$.

* $\lim_{x\to 0} \frac{x^4 - 2x^7}{5x^4 - 8x^9}$ type $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{x^4(1-2x^3)}{5x^4(1-\frac{8}{5}x^5)} = \lim_{x\to 0} \frac{1}{5}\left(\frac{1-2x^3}{1-\frac{8}{5}x^5}\right) = \frac{1}{5}$$

(Klønete å bruke L'H her!)

* $\lim_{x\to 0} \frac{1-\cos x}{\sin(x^2)}$ type $\frac{0}{0}$

L'H : $\lim_{x\to 0} \frac{\sin x}{\cos(x^2)\cdot 2x}$ type $\frac{0}{0}$

L'H (to ganger) $\lim_{x\to 0} \frac{\cos x}{-\sin(x^2)\cdot(2x)^2 + 2\cos(x^2)} = \frac{1}{2}$.