Uendelige følger og rekker

En følge $\{x_n\}$ konvergere til L

hvis $\lim_{n\to\infty} x_n = L$

Vi sier at $\{x_n\}$ konvergerer hvis det finnes en L s.a. $\{x_n\}$ konvergerer til L.

Hvis $\{x_n\}$ ikkje konvergerer sier vi at den divergerer.

Eks $\left\{\frac{1}{\sqrt{n}}\right\}$ $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{5}}, \ldots$

konvergerer til 0

Eks $\left\{\frac{\sin(n)}{n}\right\}$ konverger mot 0 $\left(\left|\frac{\sin n}{n}\right| < \frac{1}{n}\right)$

Opg. Konvergerer følgen $x_n = \sqrt{n^2+n} - n$?

$x_{1000} = 0.4999$ $x_6 \sim 0.4807$

$$x_n = \frac{(\sqrt{n^2+n} - n)}{1} \cdot \frac{(\sqrt{n^2+n} + n)}{(\sqrt{n^2+n} + n)} = \frac{(\sqrt{n^2+n})^2 - n^2}{\sqrt{n^2+n} + n}$$

$$= \frac{n^2+n-n^2}{\sqrt{n^2+n} + n} = \frac{n}{n\sqrt{1+\frac{1}{n}} + n} = \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

Så $\lim_{n\to\infty} x_n = \frac{1}{2}$

En rekke

$$\underbrace{x_1 + x_2 + x_3 + \cdots + x_n}_{s_n} + \cdots$$

n-te delsum s_n

konvergerer til S hvis følgen av delsummer $\{S_n\}$ konvergerer til S.

(konvergerer : finnes en slik S
divergerer : ikkje konvergere)

Eks $\quad a + ax + ax^2 + \cdots = \dfrac{a}{1-x} \quad -1 < x < 1$ (eller $a=0$)

$|x| \geq 1$: divergere $(a \neq 0)$ (Her er n-te delsum $S_n = a\dfrac{1-x^n}{1-x}$)

$a = 1 \quad x = 0.1$

$1 + 0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

$= 1.11111\ldots = 1 + \underbrace{0.111111}_{1/9} = \dfrac{10}{9}$

(siden $0.9999\ldots = 1$)

Dette stemmer med formelen: $\dfrac{1}{1-0.1} = \dfrac{1}{0.9}$

$= \underline{\dfrac{10}{9}}$

Oppg $\quad x^4 + x^7 + x^{10} + x^{13} + \cdots$ ~~geometrisk rekke.~~

Når konv. den og hva konv. den mot?

$x^4\left(1 + x^3 + (x^3)^2 + (x^3)^3 + \cdots\right)$

$= x^4 \cdot \dfrac{1}{1-x^3} \qquad |x^3| < 1 \iff \underline{|x| < 1}$

Konvergerer presis når $|x| < 1$. Summen er da $\underline{\dfrac{x^4}{1-x^3}}$

Oppg. $\quad 4x + 2x^3 + x^5 + \dfrac{x^7}{2} + \dfrac{x^9}{4} + \cdots$

Faktoren i denne geometriske rekken er $x^2/2$.

Konvergere $\iff \left|\dfrac{x^2}{2}\right| < 1 \iff |x^2| < 2 \iff \underline{|x| < \sqrt{2}}$

Summen er da $4x \cdot \dfrac{1}{1 - x^2/2} \cdot \dfrac{2}{2}$

$= \underline{\underline{\dfrac{8x}{2-x^2}}}$

Setter inn $-e^{-t}$ for x i den geometriske rekken:

$$1 - e^{-t} + e^{-2t} - e^{-3t} + e^{-4t} - + \cdots$$

$$= \frac{1}{1-(-e^{-t})} = \frac{1}{1+e^{-t}} = \frac{e^{t}}{e^{t}+1}$$

for $|-e^{-t}| < 1 \iff \underline{t > 0}$

Vi gir eksempler hvor vi integrerer rekker ledd for ledd. (jamfør 17.85 i boka)

$$\int_0^x (1 + t + t^2 + \cdots + t^n + \cdots)\, dt = \int_0^x \frac{1}{1-t}\, dt$$

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \frac{x^{n+1}}{n+1} + \cdots = -\ln|1-t|\Big|_0^x$$

$$= -\ln|1-x|$$

$$-1 \le x < 1$$

Setter inn $x = -1$

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + - \cdots = -\ln 2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - + \cdots = \underline{\ln 2}$$

Eks $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$

Starter med $1 - t^2 + t^4 - t^6 + \cdots = \frac{1}{1+t^2}$ $|t| < 1$

Integrerer leddvis fra 0 til x:

$$\int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt = \int_0^x \frac{1}{1+t^2}\, dt$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - = \arctan t\,\Big|_0^x$$

$$= \arctan(x)$$

$|x| \le 1.$

Setter $x = 1$. $\arctan 1 = \pi/4$

$$\underline{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \quad = \underline{\underline{1}}$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \qquad \text{delbrøksoppspalting}$$

n-te delsum: $S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$$+ \cdots \quad \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = 1.$$

Eks $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergere $\left(< 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < 2 \right)$

Faktum: $\quad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \underline{\frac{\pi^2}{6}}$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \cdots$$

leddene $\frac{1}{\sqrt{n}}$ går mot 0 når $n \to \infty$

men rekken divergerer.

$$S_n = 1 + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}} \geq \text{antall ledd} \cdot \text{minste ledd}$$

$$n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}$$

så $S_n \geq \sqrt{n}$ for alle n.

Derfor divergerer rekken.