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16.2 Delvis integrasjon

① Produktregelen $\qquad U, V$ deriverbare funksjoner

$$(U\cdot V)' = U'\cdot V + U\cdot V'$$

$$\int (U'\cdot V + U\cdot V')\,dx = U\cdot V + C$$

$$\int U'\cdot V\,dx + \int U\cdot V'\,dx = U\cdot V + C$$

DELVIS INTEGRASJON

$$\boxed{\int U'\cdot V\,dx = U\cdot V - \int U\cdot V'\,dx}$$

<u>Eks</u> $\quad \int x\cdot \sin x\,dx$ $\qquad$ La $\sin x = U'$, $x = V$

Velger $U = -\cos x$

$V' = (x)' = 1$

Delvis integrasjon: $\int x\sin x\,dx = (-\cos x)\cdot x - \int (-\cos x)\cdot 1\,dx$

$$= -x\cos x + \int \cos x\,dx$$

$$\int x\sin x\,dx = \underline{-x\cos x + \sin x + C}$$

Deriverer svaret:

(2) $(-x\cos x + \sin x + c)'$

$= -((x)' \cdot \cos x + x(\cos x)') + (\sin x)' + (c)'$

$= -\cos x + x\sin x + \cos x + 0$

$= x\sin x.$

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eks $\int \overset{v}{x^2} \overset{u'}{\cos x}\, dx$ Velger $u = \sin x$

$= x^2 \sin x - \int 2x \cdot \sin x\, dx$

$= x^2\sin x - 2\int x\sin x\, dx$

$= x^2\sin x - 2(-x\cos x + \sin x) + c$

$= \underline{x^2\sin x + 2x\cos x - 2\sin x + c}.$

oppg $\int \underbrace{x}_{v}\underbrace{e^x}_{u'}\, dx$ $u = e^x$

$= x \cdot e^x - \int 1 \cdot e^x\, dx$

$= xe^x - e^x + c$

$= \underline{e^x(x-1) + c}$

eks 3

$$\int \ln|x|\,dx = \int \underset{u'}{1} \cdot \underbrace{\ln|x|}_{v}\,dx$$

Velger $u = x$

$v' = (\ln|x|)' = \frac{1}{x}$

$$\int \ln|x|\,dx = x\ln|x| - \int x \cdot \frac{1}{x}\,dx$$

$$= x\ln|x| - \int 1\,dx$$

$$= x\ln|x| - x + c$$

$$= \underline{x(\ln|x| - 1) + c}$$

$$\int \overset{u'}{x^4}\,\overset{v}{\ln|x|}\,dx$$

$u = \frac{x^5}{5}$

$v' = \frac{1}{x}$

$$= \frac{x^5}{5}\ln|x| - \int \frac{x^5}{5} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^5}{5}\ln|x| - \frac{1}{5}\int x^4\,dx$$

$$= \frac{x^5}{5}\ln|x| - \frac{1}{5} \cdot \frac{1}{5}x^5 + c$$

$$= \underline{\frac{x^5}{5}\left(\ln|x| - \frac{1}{5}\right) + c}\,.$$

1. forslag $\int (\ln x)^2 \, dx$

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$$= \int \overset{U'}{\ln x} \cdot \overset{V}{\ln x} \, dx$$

$$= x(\ln x - 1) \cdot \ln x - \int x(\ln x - 1) \cdot \frac{1}{x} \, dx$$

$$= x(\ln x - 1)\ln x - \int \ln x - 1 \, dx$$

$$= x(\ln x - 1)\ln x - x(\ln x - 1) + x + c$$

$$= \underline{x(\ln x)^2 - 2x\ln x + 2x + c}$$

2. forslag

$$\int \overset{U'}{1} \cdot \overset{V}{(\ln x)^2} \, dx$$

$U = x$

$V' = 2\ln x \cdot \left(\frac{1}{x}\right)$

$$= x \cdot (\ln x)^2 - \int x \cdot (2\ln x) \cdot \frac{1}{x} \, dx$$

$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$$= x(\ln x)^2 - 2\, x(\ln(x) - 1) + c$$

$$= \underline{x(\ln x)^2 - 2x\ln x + 2x + c}$$

eksempel

⑤ $$\int \underbrace{e^{2x}}_{u'} \underbrace{\sin(3x)}_{v}\, dx$$

$$U = \frac{e^{2x}}{2}$$

$$V' = 3\cdot\cos 3x$$

$$= \frac{1}{2}e^{2x}\cdot \sin 3x \quad - \int \frac{1}{2}e^{2x}\cdot 3\cdot\cos(3x)\,dx$$

$$= \frac{1}{2}e^{2x}\sin(3x) \quad - \frac{3}{2}\int \underbrace{e^{2x}}_{u'} \underbrace{\cos(3x)}_{v}\, dx$$

$$U = \frac{e^{2x}}{2}$$

$$V' = -3\sin(3x)$$

$$= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos(3x) \quad - \int \frac{e^{2x}}{2}(-3\sin(3x))dx\right]$$

$$= \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos 3x \quad - \left(\frac{3}{2}\right)^2 \int e^{2x}\sin(3x)\,dx$$

Derfor er

$$\left(1+\left(\frac{3}{2}\right)^2\right)\int e^{2x}\sin(3x)\,dx = \frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos 3x + C$$

$$\int e^{2x}\sin(3x)\,dx =$$

$$\frac{1}{\left(1+\left(\frac{3}{2}\right)^2\right)}\left(\frac{1}{2}e^{2x}\sin(3x) - \frac{3}{4}e^{2x}\cos(3x)\right) + C$$

Vi utfører delvis integrasjon to ganger og ender opp med det opprinnelige integralet (med en koeffisient ulik 1).

oppg. $\int \ln\left(|3x-2|^3\right) dx$

6 $= \int 3 \cdot \ln|3x-2|\, dx$

substitusjon $u = 3x-2 \qquad u' = 3$

$$\int \ln|3x-2|^3 dx = \int u' \cdot \ln|u|\, dx$$

$$= \int 1 \cdot \ln|u|\, du$$

$$= u \ln|u| - \int u \cdot \frac{1}{u} du$$

$$= u\ln|u| - u + c$$

$$= u(\ln|u| - 1) + c$$

$$= \underline{(3x-2)(\ln(3x-2) - 1) + c}$$

eks $\int (3x-1)e^{2x+1} dx$

Prøver med substitusjonen $u = 2x+1$

$u' = 2$

$$x = \frac{u-1}{2}$$

så $3x-1 = \frac{3}{2}(u-1) - 1 = \frac{3u-5}{2}$

$$\int \frac{3u-5}{2} e^u \cdot \overbrace{\frac{1}{2} du}^{(dx)}$$

$$= \left(\frac{1}{2}\right)^2 \int (3u-5)e^u du$$

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$$= \left(\frac{1}{2}\right)^2 \left[-5\int e^u\,du + 3\int u e^u\,du \right]$$

delvis integrasjon

$$= \left(\frac{1}{2}\right)^2 \left[-5e^u + 3\left(ue^u - e^u\right) \right] + c$$

$$= \frac{1}{4}\left(3ue^u - 8e^u\right) + c$$

$$= \frac{1}{4}\left[3(2x+1)e^{2x+1} - 8e^{2x+1}\right] + c$$

$$= \frac{1}{4}\left[(6x+3-8)e^{2x+1}\right] + c$$

$$= \underline{\frac{6x-5}{4}e^{2x+1} + c}$$

oppg. $\int (6x-1)\cos(3x)\,dx$

Substitusjon $u = 3x$ $\quad u' = 3$

$6x - 1 = 2u - 1$ $\quad du = 3dx$

$\frac{1}{3}du$ $\quad dx = \frac{1}{3}du$

$$\int (2u-1)\cos(u)\,dx$$

$W = \sin u$

$$= \frac{1}{3}\int \underbrace{(2u-1)}_{v}\underbrace{\cos u}_{w'}\,du$$

delvis integrasjon

$$= \frac{1}{3}\left[(2u-1)\sin u - \int 2\cdot\sin u\,du\right]$$

$$= \frac{1}{3}\left[(2u-1)\sin u - 2(-\cos u)\right] + c$$

$$= \underline{\frac{1}{3}(6x-1)\sin(3x) + \frac{2}{3}\cos(3x) + c}$$

Vi kan også regne ut de to siste integralen uten å bruke substitusjon først →

⑧ $\int \overset{v}{(3x-1)} \overset{u'}{e^{2x+1}}\,dx$ $\qquad u = \frac{1}{2}e^{2x+1}$, $\quad v' = 3$

$$= (3x-1)\frac{1}{2}e^{2x+1} - \int 3\cdot\frac{1}{2}e^{2x+1}\,dx$$

$$= \frac{(3x-1)}{2}e^{2x+1} - \frac{3}{2}\left(\frac{e^{2x+1}}{2}\right) + c$$

$$= \underline{\frac{(6x-5)}{4}e^{2x+1} + c}$$

Vi ser at det er enklere å utføre delvis integrasjon direkte, uten å "forenkle" ved å bruke lineær substitusjon først.

Det gjelder også den siste oppgaven.

$\int \overset{v}{(6x-1)} \overset{u'}{\cos(3x)}\,dx$ $\qquad u = \frac{\sin 3x}{3}$

$$= (6x-1)\frac{\sin 3x}{3} - \int 6\,\frac{\sin 3x}{3}\,dx$$

$$= \underline{\left(\frac{6x-1}{3}\right)\sin(3x) + \frac{2\cos 3x}{3} + c}$$