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(1) Eulers formel $e^{iz} = \cos z + i \sin z.$

Vi bruker denne til å finne formlene for dobling av vinklene og mer generelt sum av to vinkler

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

$$= e^{ix+iy} = e^{ix} \cdot e^{iy}$$

$$= (\cos x + i \sin x)(\cos y + i \sin y)$$

$$= \cos x \cdot \cos y + \underbrace{i \cdot i}_{-1} \sin x \cdot \sin y + i \cos x \sin y + i \sin x \cos y$$

$$= (\cos x \cos y - \sin x \sin y) + i(\cos x \sin y + \sin x \cos y)$$

Derfor er $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \cos x \sin y + \sin x \cdot \cos y$.

Binomialrekken

Generaliserer binomialkoeffisienten

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$ $n \geq 1$ heltall

$$\boxed{(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n}$$ $|x| < 1$

Hvorfor? $\left(\left((1+x)^\alpha\right)^{(n)} = \left(\alpha(1+x)^{\alpha-1}\right)^{(n-1)}\right.$

$$= \left(\alpha(\alpha-1)(1+x)^{\alpha-2}\right)^{(n-2)} = \cdots$$

$$= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$

Så $$\left.\frac{\left((1+x)^\alpha\right)^{(n)}}{n!}\right|_{x=0} = \underline{\binom{\alpha}{n}}$$

$\alpha = -1$: $(1+x)^{-1} = \dfrac{1}{1+x}$

② $\binom{-1}{0} = 1$ $\qquad \binom{-1}{n} = \dfrac{(-1)(-2)(-3)\cdots\overbrace{(-1-n+1)}^{-n}}{n!}$

$$\binom{-1}{n} = (-1)^n \frac{n!}{n!} = (-1)^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + - \cdots$$

(Det samme som vi får fra den geometriske rekke $\dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n$)

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α positivt heltall N. Da er binomialrekken endelig $\qquad \binom{N}{n} = 0 \qquad n > N$

$$(1+x)^N = \sum_{n=0}^{N} \binom{N}{n} x^n$$

$\binom{2}{0} = \binom{2}{2} = 1$

$\binom{2}{1} = 2$

$N=2$ $\qquad (1+x)^2 = 1 + 2x + x^2$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

$\binom{3}{0} = \binom{3}{3} = 1$

$\binom{3}{1} = \binom{3}{2} = \dfrac{3!}{1!\cdot 2!} = 3$

Hva er $(1+x)^7$?

$\alpha = -\frac{1}{2}$ $\qquad (1+x)^{-1/2} = \dfrac{1}{\sqrt{1+x}} = \sum\limits_{n=0}^{\infty} \binom{-1/2}{n} x^n$

$\binom{-1/2}{0} = 1$, $\qquad \binom{-1/2}{n} = \dfrac{(-1/2)(-1/2-1)(-1/2-2)\cdots\overbrace{(-1/2-n+1)}^{-\frac{2n-1}{2}}}{n!}$

$$= (-1)^n \left(\frac{1}{2}\right)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}\cdot x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + - \cdots$$

(3) $\frac{1}{\sqrt{0.9}} = \frac{1}{\sqrt{1-\frac{1}{10}}} = 1 - \frac{1}{2}\left(-\frac{1}{10}\right) + \frac{3}{8}\left(\frac{-1}{10}\right)^2 - \frac{5}{16}\left(\frac{-1}{10}\right)^3 + \cdots$

$= 1 + \frac{1}{20} + \frac{3}{800} + \frac{5}{16000} + \cdots$

$= 1 + 0.05 + 0.00375 + \sim \frac{1}{3000}$

$\sim \begin{array}{l} 1.05375 \\ 0.00033\ldots \end{array} = \underline{1.05408}\cancel{33}$

Eksakt $1.0540925\ldots$

7.4 Potensrekker

Potensrekke $\sum_{n=0}^{\infty} a_n x^n$ om 0

$\sum_{n=0}^{\infty} a_n (x-a)^n$ om a.

Resultat Det finnes en $R \geq 0$ (eller ∞)

slik at $\sum_{n=0}^{\infty} a_n x^n$ konvergerer $|x| < R$

divergerer $|x| > R$

For hver x, $|x| < R$ får vi tilordnet summen av $\sum_{n=0}^{\infty} a_n x^n$. Dette er en funksjon.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Vi kan lage funksjon ved å sette opp potensrekker.

Derivasjon og integrasjon av potensrekker utføres leddvis

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

f deriverbar og $f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$

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$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

Potensrekker er kontinuerlig (også på randen $x=-R$, $x=R$ hvis den konvergerer der.)

Eksempel $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{1}{n!}(n \cdot x^{n-1})$$

$n! = 1 \cdot 2 \cdots (n-1) \cdot n$ så $$\frac{n}{n!} = \frac{\cancel{n}}{1\cdot 2 \cdots (n-1)\cdot \cancel{n}} = \frac{1}{(n-1)!}$$

La $(n-1=m)$

$$(e^x)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$$

—

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - + = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$(\cos x)' = \sum_{n=1}^{\infty} (-1)^n \frac{2n\, x^{2n-1}}{(2n)!}$$

$n-1=m$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!} = - \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

$$= -\sin x$$

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$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

(alternativt bruk binomialformelen)

(5) Taylor rekken til (funksjonen gitt ved en) potensrekke er potensrekken selv.

n-te ledd i Taylor rekken (om 0) er $\dfrac{f^{(n)}(0)}{n!}$

$$\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)^{(n)} \Big|_{x=0}$$

$$= n!\, a_n + (n+1)\cdot n \cdots 2 \cdot a_{n+1} \cdot x + \cdots \Big|_{x=0}$$

$$= n!\, a_n .$$

Så $\dfrac{f^{(n)}(0)}{n!} = \dfrac{n!\, a_n}{n!} = a_n$

når $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Eks $-\ln|1-x| = \int_0^x \frac{1}{1-x}\, dx = -\ln|1-x|\Big|_0^x$

$= -\ln|1-x|$

$$\ln|1-x| = -\int_0^x \sum_{n=0}^{\infty} x^n\, dx$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\Big|_0^x = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\ln|1-x| = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \qquad |x|<1$$

$x = \frac{1}{2}$ $\quad \ln|1-\frac{1}{2}| = \ln(\frac{1}{2}) = -\ln 2$

$$\ln 2 = \frac{1}{2} + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} + \frac{(1/2)^4}{4} + \cdots$$

⑥ Vi kan addere, skalere og multiplisere potensrekker

$f(x) = \sum a_n x^n \qquad g(x) = \sum b_n x^n$

$c\, f(x) = \sum (c \cdot a_n) x^n$ skalering

$f(x) + g(x) = \sum (a_n + b_n) x^n$ addisjon

$$f(x) \cdot g(x) = \sum_{n,m} a_n x^n \cdot b_m \cdot x^m$$

$$= \sum_{n,m} a_n \cdot b_m \, x^{n+m}$$

$$= \sum c_k x^k$$ multiplikasjon

hvor $c_k = \sum_{n+m=k} a_n b_m$

Eks.

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$= \left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{m=0}^{\infty} x^m\right)$$

$$= \sum_{k=0}^{\infty} c_k \cdot x^k = \underline{\sum_{k=0} (k+1)\, x^k}$$

$$c_k = \sum_{n+m=k} \overset{=1}{a_n} \cdot \overset{=1}{b_m} = \sum_{n+m=k} 1 = k+1$$